Announcements

1) Error on take-home midterm! Need a vector space of odd dimension!

Recall! orthogonal projection

V an inner product space, $W \subseteq V$, W is spanned by either a finite or countable set of vectors $\{x_i\}$. Gram-Schmidt gives us an orthogonal set $\{z_i\}$ with

$$\mathrm{span}(\{z_i\}) = W$$

$$z_i = w_i - \sum_{k=1}^{i-1} \frac{\langle w_i, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

If $x \in V$, we defined the orthogonal projection Px to be (W finite-dimensional, $\dim(W) = n$)

$$Px = \sum_{k=1}^{n} \frac{\langle x, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

Observe that if $x, y \in V$,

$$P(x+y)$$

$$= \sum_{k=1}^{n} \frac{\langle x+y, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

$$= \sum_{k=1}^{n} \frac{\langle x, z_k \rangle + \langle y, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

by inner-product properties

$$= \sum_{k=1}^{n} \frac{\langle x, z_k \rangle}{\langle z_k, z_k \rangle} z_k + \sum_{k=1}^{n} \frac{\langle y, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

$$= Px + Py$$

Also, if $\alpha \in \mathbb{R}$ or $\mathbb{C}$,

$x \in V$,

$$P(\alpha x) = \sum_{k=1}^{n} \frac{\langle \alpha x, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

$$= \sum_{k=1}^{n} \alpha \frac{\langle x, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

by properties of the inner product

$$= \alpha \sum_{k=1}^{n} \frac{\langle x, z_k \rangle}{\langle z_k, z_k \rangle} z_k$$

$$= \alpha P x$$

We see that

$$P(x+y) = Px + Py$$

$$P(\alpha x) = \alpha Px$$

$$\forall\, x, y \in V,\ \alpha \in \mathbb{R} \text{ or } \mathbb{C}.$$

We now look at general maps between vector spaces with this property!

Linear Transformations

(Chapter 2)

$\mathbb{R}^2$ vs $\mathbb{C}$

As a vector space over $\mathbb{R}$, $\mathbb{C}$ has dimension 2.

As a vector space over $\mathbb{R}$, $\mathbb{R}^2$ has dimension 2.

These appear to be no different as vector spaces over $\mathbb{R}$. How do we make this precise?

Definition: (linear transformation)

Let V and W be vector spaces over the same field $\mathbb{F}$.

A function $T: V \to W$ is called a linear transformation if $\forall\ x, y \in V$, $\alpha \in \mathbb{F}$

1) $T(x+y) = T(x) + T(y)$

2) $T(\alpha x) = \alpha T(x)$

We've already seen that the orthogonal projection $x \mapsto Px$ gives a linear transformation of an inner product space V onto a finite-dimensional subspace W.

Example 1: ($\mathbb{F}$)

Fix $\alpha \in \mathbb{F}$. Define,

for all $\beta \in \mathbb{F}$,

$T_\alpha : \mathbb{F} \to \mathbb{F}$

$$\boxed{T_\alpha(\beta) = \alpha \cdot \beta.}$$

I claim that T_α is linear $\forall\ \alpha \in \mathbb{F}$.

1) Let $\beta, \gamma \in \mathbb{F}$.

$$T_\alpha(\beta+\gamma) = \alpha \cdot (\beta+\gamma)$$

$$= \alpha \cdot \beta + \alpha \cdot \gamma$$

(field distributivity)

$$= T_\alpha(\beta) + T_\alpha(\gamma) .$$

2) Let $\gamma, \beta \in \mathbb{F}$.

$$T_\alpha(\gamma\beta) = \alpha(\gamma\beta)$$

$$= (\alpha \cdot \gamma)\beta$$

(associativity)

$$= (\gamma \cdot \alpha)\beta$$

(commutativity)

$$= \gamma \cdot (\alpha\beta)$$

(associativity)

$$= \gamma \cdot T_\alpha(\beta)$$

This shows T_α is linear $\forall\ \alpha \in \mathbb{F}$.

In particular, if $\alpha = 1_{\mathbb{F}}$, we get the identity map $\beta \mapsto \beta$ on $\mathbb{F}$, and if $\alpha = 0_{\mathbb{F}}$, we get the zero map $\beta \mapsto 0_{\mathbb{F}}$.

Claim: If $T : \mathbb{F} \to \mathbb{F}$ is linear, then $\exists\, \alpha \in \mathbb{F}$, $T = T_\alpha$.

proof: Let $\alpha = T(1_{\mathbb{F}})$.

Then $\forall \beta \in \mathbb{F}$,

$$T_\alpha(\beta) = \alpha\beta$$
$$= \beta\alpha$$

(commutativity)

equal

$$T(\beta) = T(\beta \cdot 1_{\mathbb{F}})$$
$$= \beta T(1_{\mathbb{F}})$$

(by linearity of T)

$$= \beta\alpha$$

So $T_\alpha = T$ for all $\beta \in \mathbb{F}$, and are therefore the same function!

This says every linear transformation from a field to itself is scalar multiplication.

Example 2: $M_{n \times m}(\mathbb{F})$.

Given a matrix

$$A = (\alpha_{i,j})_{i=1,\, j=1}^{n \quad m},$$

A defines a map from $\mathbb{F}^m$ to $\mathbb{F}^n$ by if $x_1, \ldots, x_m \in \mathbb{F}$,

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m} a_{1,j} x_j = y_1 \in \mathbb{F} \\ \sum_{j=1}^{m} a_{2,j} x_j = y_2 \in \mathbb{F} \\ \vdots \\ \sum_{j=1}^{m} a_{n,j} x_j = y_n \in \mathbb{F} \end{bmatrix}$$

Let's check this is linear.

1) Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$, $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$

be elements of $\mathbb{F}^m$.

Then

$$A(x+z) = A\left(\begin{bmatrix} x_1+z_1 \\ x_2+z_2 \\ \vdots \\ x_m+z_m \end{bmatrix}\right)$$

$$= \begin{bmatrix} \sum_{j=1}^{m} \alpha_{1,j}(x_j+z_j) \\ \vdots \\ \sum_{j=1}^{m} \alpha_{n,j}(x_j+z_j) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{m} \alpha_{1,j}x_j + \sum_{j=1}^{m} \alpha_{1,j}z_j \\ \vdots \\ \sum_{j=1}^{m} \alpha_{n,j}x_j + \sum_{j=1}^{m} \alpha_{n,j}z_j \end{bmatrix}$$

$$= Ax + Az$$

b) Let $\alpha \in \mathbb{F}$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{F}^m.$$

Then $A(\alpha x) = A\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}\right)$

$$= \begin{bmatrix} \sum_{j=1}^{m} a_{1,j} \alpha x_j \\ \vdots \\ \sum_{j=1}^{m} a_{n,j} \alpha x_j \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{m} \alpha a_{1,j} x_j \\ \vdots \\ \sum_{j=1}^{m} \alpha a_{n,j} x_j \end{bmatrix}$$ Commutativity

$$= \alpha \begin{bmatrix} \sum_{j=1}^{m} a_{1,j} x_j \\ \vdots \\ \sum_{j=1}^{m} a_{n,j} x_j \end{bmatrix} = \alpha A(x)$$

So A is linear,

$$A: \mathbb{F}^m \to \mathbb{F}^n.$$